SEPARATION DOES NOT IMPLY REPLACEMENT IN FIRST ORDER GENERIC EXTENSIONS FOR CLASS FORCING

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Abstract. We continue the investigation of the relationship between the axiom schemes of separation and replacement in generic extensions for class forcing over second order models of Gödel-Bernays set theory GB, that was started in [HKS]. We construct a notion of class forcing $\mathbb{P}$, for a countable and transitive second order model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB, that satisfies the forcing theorem and preserves all axioms of the separation scheme to some generic first order extension $M[G]$ without predicates, however fails to preserve replacement to this extension, by adding a definable cofinal function from ω to Ord^M over $M[G]$. The construction builds on the related [HKS, Theorem 7.7] and answers [HKS, Question 9.3]. Moreover, we show that the consistency strength of the assumption used in [HKS, Theorem 7.7] and in the construction in this paper lies strictly between a weakly compact and a measurable cardinal.

As an auxiliary result, that may be of independent interest, we show that every notion of class forcing, with the property that every new set is already added by some set-sized complete subforcing, satisfies the forcing theorem, and we provide a combinatorial characterization of the above property, that we call the *set reduction property*. This is a generalization of results on the set decision property in [HKS, Section 6].

1. Introduction

Unlike in set forcing, where a first order model of set theory together with a generic filter for a notion of forcing in that model yield a unique generic extension of that model, in class forcing there are several possible candidates for what could be considered the relevant generic extension, depending on which predicates one allows for. This was studied in some detail in [HKS, Section 2], and we now provide a short review before we are able to properly state the main results of this paper. Readers who are not familiar with the basic setup used in [HKL$^+$] or [HKS] may want to read through the review thereof, that is provided in Section 2, before continuing to read this introduction.

If the ground model is of the form $\mathbb{M} = \langle M, \mathcal{C}\rangle$ and G is an $\mathbb{M}$-generic filter for some notion of class forcing for $\mathbb{M}$, as in set forcing one may simply consider $M[G]$ to be the generic extension, which we call the *generic set-extension*. Alternatively, one may want to include a predicate for the generic filter G and all the predicates from $\mathcal{C}$, that were available already in the ground model. Note that $\mathcal{C}$ in particular includes a predicate for the ground model M. This yields the *generic class extension* $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$, where $\mathcal{C}[G] = \{C[G] \mid C \in \mathcal{C}\}$ is the collection of all G-evaluations of class names in $\mathcal{C}$. More generally, one may want to consider generic extensions of the form $\mathbb{N} = \langle M[G], \mathcal{D}\rangle$ with $\{M, G\} \subseteq \mathcal{D} \subseteq \mathcal{C}[G]$, with $\mathcal{D}$ closed under definability over $\mathbb{N}$. We call such extensions *generic class pseudo-extensions*. We refer to any of the above extensions as *generic extensions*.

In [HKS, Theorem 5.2], it is shown that if $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- so that $\mathcal{C}$ contains a set-like wellorder, $\mathbb{P} \in \mathcal{C}$ is a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem and G is $\mathbb{P}$-generic over $\mathbb{M}$, then in the generic class extension $\mathbb{M}[G]$, separation implies replacement.

This is contrasted in [HKS, Theorem 7.7], where it is shown that assuming the consistency of a measurable cardinal, the above may fail for generic class pseudo-extensions. This left open the very

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natural question (see [HKS, Question 9.3]) whether this is also the case for generic set-extensions, and in the main result of this paper, we will answer this question positively.

Theorem 1.1. *Assuming the consistency of a measurable cardinal, there is a second order model $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GBC}$ and a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem, so that in every generic set-extension $M[G]$ of $\mathbb{M}$ by an $\mathbb{M}$-generic filter G for $\mathbb{P}$, separation holds while replacement fails.*

The forcing witnessing [HKS, Theorem 7.7] is a class sized version of Prikry forcing, that adds a cofinal function from ω to Ord^M while not adding any new sets and satisfying a Prikry property that implies the preservation of separation. In a nutshell, the basic idea of the adaption of this forcing that we will present in this paper is to additionally code this cofinal function into the GCH pattern and thus make it definable over $M[G]$. The property of not adding new sets is replaced by the property that every new set is added by some set-size complete subforcing, and the Prikry property is replaced by what we call a *Prikry reduction property*, that is the existence of a notion of direct extension that reduces the decision about a given statement to some set-sized complete subforcing.

2. Basic Setup

We will work with the same setup as in [HKS], which we will shortly review here for the benefit of the reader. Namely, we will work with transitive second-order models of set theory, that is models of the form $\mathbb{M} = \langle M, \mathcal{C}\rangle$, where M is transitive and denotes the collection of *sets* of $\mathbb{M}$ and $\mathcal{C}$ denotes the collection of *classes* of $\mathbb{M}$.[1] We require that $M \subseteq \mathcal{C}$ and that elements of $\mathcal{C}$ are subsets of M. Classical transitive first-order models of set theory are covered by our approach when we let $\mathcal{C}$ be the collection of classes definable over $\langle M, \in\rangle$. The theories that we will be working in will be fragments of *Gödel-Bernays set theory* GB: We denote by GB^- the theory in the two-sorted language with variables for sets and classes, with the set axioms given by ZF^- with class parameters allowed in the schemata of separation and collection, and the class axioms of extensionality, foundation and first-order class comprehension (i.e. involving only set quantifiers). GB^- enhanced with the power set axiom is the common collection of axioms of GB. GBC is GB together with the axiom of global choice. By a countable transitive model of GB^-, GB or GBC, we mean a transitive second-order model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^-, GB or GBC respectively, such that both M and $\mathcal{C}$ are countable in V. Given a transitive second-order model of the form $\mathbb{M} = \langle M, \mathcal{C}\rangle$, we let $\mathrm{Def}(\mathbb{M})$ denote the collection of subsets of M that are first-order definable over M using class parameters from $\mathcal{C}$ as predicates.[2]

Fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^-. By a *notion of class forcing* (for $\mathbb{M}$) we mean a partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ such that $P, \leq_{\mathbb{P}} \in \mathcal{C}$. We will frequently identify $\mathbb{P}$ with its domain P. In the following, we also fix a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ for $\mathbb{M}$.

We call σ a $\mathbb{P}$-*name* if all elements of σ are of the form $\langle \tau, p\rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Define $M^{\mathbb{P}}$ to be the set of all $\mathbb{P}$-names that are elements of M and define $\mathcal{C}^{\mathbb{P}}$ to be the set of all $\mathbb{P}$-names that are elements of $\mathcal{C}$. In the following, we will usually call the elements of $M^{\mathbb{P}}$ simply $\mathbb{P}$-*names* and we will call the elements of $\mathcal{C}^{\mathbb{P}}$ *class* $\mathbb{P}$-*names*.

We say that a filter G on $\mathbb{P}$ is $\mathbb{P}$-*generic over* $\mathbb{M}$ if G meets every dense subset of $\mathbb{P}$ that is an element of $\mathcal{C}$. Given such a filter G and a $\mathbb{P}$-name σ, we recursively define the G-*evaluation* of σ as

$$\sigma^G = \{\tau^G \mid \exists p \in G\,[\langle \tau, p\rangle \in \sigma]\},$$

and similarly we define Γ^G for $\Gamma \in \mathcal{C}^{\mathbb{P}}$. Moreover, if G is $\mathbb{P}$-generic over $\mathbb{M}$, then we set $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$ and $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$.

Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$, where $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ is a sequence of class name parameters, $p \in \mathbb{P}$ and $\vec{\sigma} \in (M^{\mathbb{P}})^m$, we write $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\vec{\sigma}, \vec{\Gamma})$ if for every $\mathbb{P}$-generic filter G over $\mathbb{M}$ with $p \in G$, $\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G)$.

[1] Arguing in the ambient universe V, we will sometimes refer to classes of such a model $\mathbb{M}$ as sets, without meaning to indicate that they are sets of $\mathbb{M}$. In particular this will be the case when we talk about subsets of M.

[2] Note that the axiom of first-order class comprehension implies that if $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, then $\mathcal{C}$ is closed under definability (over $\mathbb{M}$), that is $\mathrm{Def}(\mathbb{M}) = \mathcal{C}$.

The following is the natural generalization of the forcing theorem to the context of class forcing over second order models of set theory (see [HKL+] for more details).

Definition 2.1. Let $\varphi \equiv \varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$ be an $\mathcal{L}_\in$-formula with class name parameters $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$.

(1) We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$ if

$$\{\langle p, \sigma_0, \ldots, \sigma_{m-1}\rangle \in P \times (M^{\mathbb{P}})^m \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})\} \in \mathcal{C}.$$

(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$ if for all $\sigma_0, \ldots, \sigma_{m-1} \in M^{\mathbb{P}}$, for all $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with

$$\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G),$$

there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})$.

(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$ if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

Note that in class forcing, the forcing theorem may fail even for atomic formulae ([HKL+, Theorem 7.3]). A crucial result is that if the definability lemma holds for one atomic formula, then the forcing theorem holds for each $\mathcal{L}_\in$-formula with class name parameters (see [HKL+, Theorem 4.3]).

3. The Set Reduction Property

In [HKS, Section 6], we defined a combinatorial principle for a notion of class forcing $\mathbb{P}$, that we called the *set decision property*, and we showed that it is equivalent to $\mathbb{P}$ not adding new sets, and moreover that it implies the forcing theorem to hold for $\mathbb{P}$. In this section, we generalize the results from [HKS, Section 6] – we introduce the weaker *set reduction property*, and show that it is equivalent to the property that every new set added by $\mathbb{P}$ is already added by some set-size complete subforcing of $\mathbb{P}$, and moreover that it still implies the forcing theorem to hold for $\mathbb{P}$.

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Let $\dot{G}$ denote the canonical $\mathbb{P}$-name for the generic filter. Given conditions p and q in $\mathbb{P}$, we write $p \leq^*_{\mathbb{P}} q$ iff $\forall r \leq_{\mathbb{P}} p\ r \parallel_{\mathbb{P}} q$ (equivalently, $p \Vdash_{\mathbb{P}} q \in \dot{G}$). Note that if $\mathbb{P}$ is separative, then $p \leq^*_{\mathbb{P}} q$ if and only if $p \leq_{\mathbb{P}} q$.

We let $\mathbb{Q} \lessdot\!\!\circ\, \mathbb{P}$ denote the statement that $\mathbb{Q}$ is a set-sized complete subforcing of $\mathbb{P}$. We say that *every new set added by* $\mathbb{P}$ *is added by a set-sized complete subforcing of* $\mathbb{P}$ if whenever G is $\mathbb{P}$-generic over $\mathbb{M}$ and $x \in M[G] \setminus M$, then there is $\mathbb{Q} \lessdot\!\!\circ\, \mathbb{P}$ such that x is already an element of the induced $\mathbb{Q}$-generic extension $M[\bar{G}]$ of M, where $\bar{G} = G \cap \mathbb{Q}$. We will show that any $\mathbb{P}$ with this property satisfies the forcing theorem, generalizing our result on the set decision property from [HKS, Section 6], and also generalizing a classical result of Zarach ([Zar73]), where he showed that any notion of forcing that is the Ord^M-length union of complete set-sized subforcings satisfies the forcing theorem.[3] Given $p \in \mathbb{P}$ and $\mathbb{Q} \lessdot\!\!\circ\, \mathbb{P}$, let $\mathbb{Q}^{\parallel p}$ be the set of conditions in $\mathbb{Q}$ that are compatible with p in $\mathbb{P}$.

Definition 3.1. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. We say that $\mathbb{P}$ satisfies the *set reduction property (over* $\mathbb{M}$*)* if whenever $A \subseteq \mathbb{P}$ is a set (in M) and $p \in \mathbb{P}$, then there is $q \leq_{\mathbb{P}} p$ and $\mathbb{Q} \lessdot\!\!\circ\, \mathbb{P}$ (in M) such that $(*)(A, q, \mathbb{Q})$ holds: for all $a \in A$, $\{r \in \mathbb{Q}^{\parallel q} \mid \forall s \leq_{\mathbb{P}} q, r\ (s \leq^*_{\mathbb{P}} a)$ or $\forall s \leq_{\mathbb{P}} q, r\ (s \perp_{\mathbb{P}} a)\}$ is dense in $\mathbb{Q}^{\parallel q}$.

Remark 3.2. The set decision property (see [HKS, Definition 6.1]) implies the set reduction property, as is witnessed by the trivial forcing.

Definition 3.3. Given a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$, $\sigma \in M^{\mathbb{P}}$, $\mathbb{Q} \lessdot\!\!\circ\, \mathbb{P}$ and $q \in \mathbb{P}$, we define a $\mathbb{Q}$-name $\sigma^{\mathbb{Q}}_q$, the *q-reduction of* σ *to* $\mathbb{Q}$, by recursion as follows.

$$\sigma^{\mathbb{Q}}_q = \{\langle \tau^{\mathbb{Q}}_q, r\rangle \mid r \in \mathbb{Q} \wedge \exists a\,[\langle \tau, a\rangle \in \sigma \wedge \forall s \leq_{\mathbb{P}} q, r\,(s \leq^*_{\mathbb{P}} a)]\}$$

[3] A generalization of this result in a different direction was also obtained in [HKL+, Section 6].

Definition 3.4. Given a notion of class forcing $\mathbb{P}$ and a $\mathbb{P}$-name σ, we define the conditions appearing in (the transitive closure of) σ by induction on name rank as

$$\mathrm{tc}(\sigma)=\bigcup\{\{p\}\cup\mathrm{tc}(\tau)\mid\langle\tau,p\rangle\in\sigma\}.$$

Lemma 3.5. *Suppose that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$, $q\in\mathbb{P}$ and $\mathbb{Q}\lessdot\mathbb{P}$, suppose that $(*)(A,q,\mathbb{Q})$ holds and let G be $\mathbb{P}$-generic with $q\in G$. Then for every $\sigma\in M^{\mathbb{P}}$ with $\mathrm{tc}(\sigma)\subseteq A$, $\sigma^G=(\sigma_q^{\mathbb{Q}})^{\bar G}$, where $\bar G=G\cap\mathbb{Q}$.*

Proof. We proceed by induction on the name rank of σ. Suppose that $\tau^G\in\sigma^G$, because there is $a\in G$ so that $\langle\tau,a\rangle\in\sigma$. Using $(*)(A,q,\mathbb{Q})$, we can find a condition $r\in\bar G$ such that for all $s\leq_{\mathbb{P}}q,r$, it holds that $s\leq^*_{\mathbb{P}}a$. Then $\langle\tau_q^{\mathbb{Q}},r\rangle\in\sigma_q^{\mathbb{Q}}$ and by induction, $(\tau_q^{\mathbb{Q}})^{\bar G}=\tau^G$, hence $\tau^G\in(\sigma_q^{\mathbb{Q}})^{\bar G}$. If on the other hand $(\tau_q^{\mathbb{Q}})^{\bar G}\in(\sigma_q^{\mathbb{Q}})^G$, because there is $r\in\bar G$ such that $\exists a\,\langle\tau,a\rangle\in\sigma\wedge\forall s\leq_{\mathbb{P}}q,r\ s\leq^*_{\mathbb{P}}a$, then inductively $\tau^G=(\tau_q^{\mathbb{Q}})^{\bar G}\in\sigma^G$. □

Lemma 3.6. *Every notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ with the set reduction property satisfies the forcing theorem.*

Proof. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ with the set reduction property. We show that $\{\langle p,\sigma,\tau\rangle\in M\mid p\Vdash_{\mathbb{P}}\sigma=\tau\}\in\mathcal{C}$, which suffices by [HKL$^+$, Theorem 4.3]. Fix $\mathbb{P}$-names σ and τ and let $A=\mathrm{tc}(\sigma\cup\tau)$.

Claim 3.7. $p\Vdash_{\mathbb{P}}\sigma=\tau\iff\forall q\leq_{\mathbb{P}}p\ [\exists\mathbb{Q}\lessdot\mathbb{P}\,(*)(A,q,\mathbb{Q})\to q\Vdash_{\mathbb{P}}\sigma=\tau]$.

Proof. The left to right direction is immediate. For the right to left direction, note that $D=\{q\leq_{\mathbb{P}}p\mid\exists\mathbb{Q}\lessdot\mathbb{P}(*)(A,q,\mathbb{Q})\}\in\mathcal{C}$ by first order class comprehension, and that D is dense below p as a direct consequence of the set reduction property. □

Claim 3.8. *Assume that $\mathrm{tc}(\sigma)\cup\mathrm{tc}(\tau)\subseteq A$ and $(*)(A,q,\mathbb{Q})$ holds. Then $q\Vdash_{\mathbb{P}}\sigma=\tau$ if and only if*

$$\forall r_0\in\mathbb{Q}^{\parallel q}\,\exists r_1\in\mathbb{Q}^{\parallel q}(r_1\leq_{\mathbb{Q}}r_0\wedge r_1\Vdash_{\mathbb{Q}}\sigma_q^{\mathbb{Q}}=\tau_q^{\mathbb{Q}}.)$$

Proof. For the forward direction, assume that $q\Vdash_{\mathbb{P}}\sigma=\tau$ and that $r_0\in\mathbb{Q}^{\parallel q}$. Let G be $\mathbb{P}$-generic with $r_0,q\in G$, hence $\sigma^G=\tau^G$. Let $\bar G$ denote the $\mathbb{Q}$-generic induced by G, that is $\bar G=G\cap\mathbb{Q}$. By Lemma 3.5, $(\sigma_q^{\mathbb{Q}})^{\bar G}=(\tau_q^{\mathbb{Q}})^{\bar G}$. Let $r_1\leq_{\mathbb{Q}}r_0$ be a condition in $\bar G$ forcing this. Then $r_1\in Q^{\parallel q}$ and $r_1\Vdash_{\mathbb{Q}}\bar\sigma_q=\bar\tau_q$.

For the backward direction, suppose that the right-hand side holds and let G be $\mathbb{P}$-generic with $q\in G$. Let $\bar G=G\cap\mathbb{Q}$. Take $r\in\bar G$ with $r\Vdash_{\mathbb{Q}}\sigma_q^{\mathbb{Q}}=\tau_q^{\mathbb{Q}}$. Then $\sigma^G=\tau^G$ by Lemma 3.5. Since G was arbitrary, this means that $q\Vdash_{\mathbb{P}}\sigma=\tau$. □

Note that since $\mathbb{Q}$ is a notion of set forcing, it satisfies the forcing theorem, and thus the $\mathbb{Q}$-forcing relation is definable over M. Using the above claims, it is immediate that $\{\langle p,\sigma,\tau\rangle\in M\mid p\Vdash_{\mathbb{P}}\sigma=\tau\}$ is definable over $\langle M,\mathbb{P},\leq_{\mathbb{P}}\rangle$, and is thus an element of $\mathcal{C}$. □

Lemma 3.9. *Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Then $\mathbb{P}$ has the set reduction property if and only if every new set added by $\mathbb{P}$ is added by a set-sized complete subforcing of $\mathbb{P}$.*

Proof. The forward direction is immediate by Lemma 3.5. For the backward direction, let $A\subseteq\mathbb{P}$ be a set of conditions and let $\sigma=\{\langle\check a,a\rangle\mid a\in A\}$. Assume that every new set added by $\mathbb{P}$ is added by a set-sized complete subforcing of $\mathbb{P}$. However, suppose for a contradiction that $\mathbb{P}$ does not have the set reduction property, as is witnessed by $A\in M$, i.e. there is $p\in\mathbb{P}$ such that for every $q\leq_{\mathbb{P}}p$ and every $\mathbb{Q}\lessdot\mathbb{P}$ in M, there is $a\in A$ so that

$$\bar D_{q,a}=\{r\in\mathbb{Q}^{\parallel q}\mid\forall s\leq_{\mathbb{P}}q,r\,(s\leq^*_{\mathbb{P}}a)\text{ or }\forall s\leq_{\mathbb{P}}q,r\,(s\perp_{\mathbb{P}}a)\}$$

is not dense in $\mathbb{Q}^{\parallel q}$. We want to use this assumption to find a $\mathbb{P}$-generic filter G over $\mathbb{M}$ such that σ^G does not lie in the induced $\mathbb{Q}$-generic extension for any $\mathbb{Q}\lessdot\mathbb{P}$, i.e. not every new set is added by a set-sized complete subforcing.

We enumerate all dense subclasses of $\mathbb{P}$ which are in $\mathcal{C}$ (from the outside) by $\langle D_n\mid n\in\omega\rangle$, all $\mathbb{Q}\lessdot\mathbb{P}$ by $\langle\mathbb{Q}_n\mid n\in\omega\rangle$ so that every $\mathbb{Q}\lessdot\mathbb{P}$ is enumerated unboundedly often, and we let $\langle\rho_n\mid n\in\omega\rangle$ be so that each ρ_n is a $\mathbb{Q}_n$-name for a subset of A, and so that for every $i\in\omega$, every $\mathbb{Q}_i$-name ρ is enumerated as some ρ_n.

Now we define a decreasing sequence of conditions $\langle q_n \mid n \in \omega\rangle$ below p and a sequence $\langle a_n \mid n \in \omega\rangle$ of conditions in A. Let $q_0 = p$. Given $q_n \leq_{\mathbb{P}} p$, we use our assumption to pick $a_n \in A$ such that $\bar{D}_{q_n,a_n}$ is not dense in $\mathbb{Q}_n^{\parallel q_n}$. We may thus pick $r_0 \in \mathbb{Q}_n^{\parallel q_n}$ such that no $r_1 \leq_{\mathbb{Q}_n} r_0$ lies in this set. Pick $r_1 \leq_{\mathbb{Q}_n} r_0$ in $\mathbb{Q}_n^{\parallel q_n}$ which decides whether or not $\check{a}_n \in \rho_n$. This can be done because if B is a maximal antichain, of conditions below r_0 in $\mathbb{Q}_n$ which decide whether or not $\check{a}_n \in \rho_n$, then B is also maximal below r_0 in $\mathbb{P}$, since $\mathbb{Q}_n$ is a complete subforcing of $\mathbb{P}$. In particular there must be $r_1 \in B$ which is compatible with q_n in $\mathbb{P}$.

Since $r_1 \notin \bar{D}_{q_n,a_n}$, we may now pick $\tilde{q}_n \leq_{\mathbb{P}} q_n, r_1$ such that $\tilde{q}_n \perp_{\mathbb{P}} a_n$ in case $r_1 \Vdash_{\mathbb{Q}_n} \check{a}_n \in \rho_n$, and such that $\tilde{q}_n \leq^*_{\mathbb{P}} a_n$ in case $r_1 \Vdash_{\mathbb{Q}_n} \check{a}_n \notin \rho_n$. Now take $q_{n+1} \leq_{\mathbb{P}} \tilde{q}_n$ such that $q_{n+1} \in D_n$. In the end, this constructions yields a $\mathbb{P}$-generic filter $G = \{q \in \mathbb{P} \mid \exists n \in \omega\, (q_n \leq_{\mathbb{P}} q)\}$. But since every new set added by $\mathbb{P}$ is added by a set-sized complete subforcing by assumption, and since $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \subseteq \check{A}$, there must be some $n \in \omega$ such that $M[G] \models \sigma^G = \rho_n^{\bar{G}_n}$, where $\bar{G}_n = G \cap \mathbb{Q}_n$. But either $q_{n+1} \Vdash_{\mathbb{P}} \check{a}_n \in \rho_n$ and $a_n \perp_{\mathbb{P}} q_{n+1}$, thus $a_n \notin \sigma^G$, or $q_{n+1} \Vdash_{\mathbb{P}} \check{a}_n \notin \rho_n$ and $q_{n+1} \leq^*_{\mathbb{P}} a_n$, implying that $a_n \in \sigma^G$. Thus $\sigma^G \neq \rho_n^{\bar{G}_n}$, and we have reached a contradiction. □

Putting together Lemma 3.6 and 3.9 we obtain

Corollary 3.10. *If $\mathbb{P}$ is a notion of class forcing such that every new set added by $\mathbb{P}$ is already added by a set-sized complete subforcing of $\mathbb{P}$, then $\mathbb{P}$ satisfies the Forcing Theorem.*

4. Collapses with set-sized complete subforcings

The forcing notion that we will introduce in the next section in order to prove the main result of this paper is based on a variant of a collapse forcing introduced in [HKL$^+$, Definition 2.1 (3)].

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, let $\mathrm{Col}(\omega, \mathrm{Ord})^M$ denote the notion of forcing with conditions of the form $p\colon n \to \mathrm{Ord}^M$ for $n \in \omega$, ordered by reverse inclusion. Any generic for this notion of forcing clearly gives rise to a cofinal sequence from ω to Ord^M, this forcing notion does not add any new sets (i.e. $M[G] = M$ whenever G is $\mathrm{Col}(\omega, \mathrm{Ord})^M$-generic over M) and it satisfies the forcing theorem (see [HKL$^+$, Section 2] or [HKS, Section 6]). However this notion of forcing does not have any nontrivial set-sized maximal antichains, and thus no nontrivial set-sized complete subforcings (see [HKL$^+$, Lemma 2.2]).

Now, following [HKL$^+$], we will introduce a larger forcing notion that is the union of set-sized complete subforcings, in which $\mathrm{Col}(\omega, \mathrm{Ord})^M$ lies dense and that we will make use of in the next section, namely the forcing $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$. We first need some auxiliary definitions. For each $\beta \in \mathrm{Ord}^M$, let "$\geq\beta$" be an element of M that is not an ordinal and that codes β in some simple way, for definiteness say "$\geq\beta$" $= \langle\beta, \beta\rangle$. Let $\mathrm{Ord}^M_{\geq} = \{\text{“}\geq\beta\text{”} \mid \beta \in \mathrm{Ord}^M\}$ and let $D = \mathrm{Ord}^M \cup \mathrm{Ord}^M_{\geq}$. Given $x \in D$, let $\mathrm{Ord}(x)$ be the unique γ such that either $x = \gamma$ or $x =$ "$\geq\gamma$". We define an extension relation on D, by stipulating that, for $x, y \in D$,

$$x \leq_D y \iff \begin{array}{l} x = y \in \mathrm{Ord}^M \text{ or} \\ y \in \mathrm{Ord}^M_{\geq} \text{ and } \mathrm{Ord}(x) \geq \mathrm{Ord}(y). \end{array}$$

We say that $x \parallel_D y$ iff there is $z \in D$ such that $z \leq_D x$ and $z \leq_D y$. Provided $x \parallel_D y$, let

$$x \wedge_D y = \begin{cases} x & x \in \mathrm{Ord}^M, \\ y & y \in \mathrm{Ord}^M, \\ \text{“}\geq \max\{\mathrm{Ord}(x), \mathrm{Ord}(y)\}\text{”} & \text{otherwise.} \end{cases}$$

$\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ consists of finite sequences $s = \langle s(i) \mid i < n\rangle$ from D such that for all $i < n-1$, if $s(i+1) \in \mathrm{Ord}^M$, then $s(i) \in \mathrm{Ord}^M$ and $s(i) < s(i+1)$. Given $s, t \in \mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$, let $t \leq s$ if

(1) $\mathrm{dom}(t) \supseteq \mathrm{dom}(s)$ and
(2) for every $i \in \mathrm{dom}(s)$, $t(i) \leq_D s(i)$.

Given any function s, let s^{Ord} denote $\mathrm{ran}(s) \cap \mathrm{Ord}$. If $\mathrm{ran}(s) = s^{\mathrm{Ord}}$ for $s \in \mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$, we will sometimes identify s and s^{Ord}, i.e. we will view s as a finite set, rather than a finite strictly increasing sequence of ordinals.

Any generic for $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ again clearly gives rise to a cofinal sequence from ω to Ord^M. $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ is not separative, since for example $\emptyset \not\leq \langle\text{“}\geq 0\text{”}\rangle$, however every extension of $\emptyset$

is compatible to $\langle``{\geq}0"\rangle$. Moreover since $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$ is the union $\bigcup_{\alpha\in\mathrm{Ord}^M}\mathrm{Col}^{\alpha}_{\geq}(\omega,\mathrm{Ord})^M$ of an Ord^M-length increasing sequence of set-sized complete subforcings $\mathrm{Col}^{\alpha}_{\geq}(\omega,\mathrm{Ord})^M=\{p\in\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M\mid\forall i\in\mathrm{dom}(p)\ \mathrm{Ord}(p(i))<\alpha\vee p(i)=``{\geq}\alpha"\}$, each of which is atomic, it does not add any new sets and thus satisfies the forcing theorem by [HKS, Corollary 6.6].

In the next chapter, we will make use of an even finer collection of set-sized complete subforcings of $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$. For $n\in\omega$ and $\alpha\in\mathrm{Ord}$, let $\mathrm{Col}^{n,\alpha}_{\geq}(\omega,\mathrm{Ord})^M=\{p\in\mathrm{Col}^{\alpha}_{\geq}(\omega,\mathrm{Ord})^M\mid\mathrm{dom}(p)=n\}$.

Lemma 4.1. *For $n\in\omega$ and $\alpha\in\mathrm{Ord}$, $\mathrm{Col}^{n,\alpha}_{\geq}(\omega,\mathrm{Ord})^M$ is a set-sized complete subforcing of $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$.*

Proof. We may assume that $n>0$. Let A be a maximal antichain of $\mathrm{Col}^{n,\alpha}_{\geq}(\omega,\mathrm{Ord})^M$ and let $p\in\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$. Consider the condition $\bar p\in\mathrm{Col}^{n,\alpha}_{\geq}(\omega,\mathrm{Ord})^M$ which is obtained from $p\restriction n$ by replacing $p(i)$ by "$\geq\alpha$" whenever $p(i)\leq_D$ "$\geq\alpha$". Since A is a maximal antichain, there is $a\in A$ such that a and $\bar p$ are compatible. Let $\bar q\in\mathrm{Col}^{n,\alpha}_{\geq}(\omega,\mathrm{Ord})^M$ be a common strengthening of $\bar p$ and of a. Let q be the sequence with $\mathrm{dom}(q)=\mathrm{dom}(p)$, where for $i<n$,

$$q(i)=\bar q(i)\wedge_D p(i),$$

and for $i\geq n$,

$$q(i)=p(i).$$

Making use of the respective properties of $\bar q$ and of p, it is immediate to observe that whenever $q(i+1)\in\mathrm{Ord}^M$, then $q(i)\in\mathrm{Ord}^M$ and $q(i)<q(i+1)$. Thus $q\in\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$ and $q\leq p,a$. Hence A is a maximal antichain in $\mathrm{Col}_{\geq}(\omega,\mathrm{Ord})^M$. □

5. Class Prikry Forcing with GCH Coding

In [HKS, Section 7], we introduced a class variant of Prikry Forcing in order to show that separation does not necessarily imply replacement in generic class pseudo-extensions, contrasting the situation in generic class extensions (see [HKS, Theorem 5.2]). We now want to consider generic set extensions.

For this purpose, over particular models of GBC, we define yet another class sized variant of Prikry Forcing, where we additionally code the Prikry sequence into the GCH pattern (we assume the GCH in the ground model) and that incorporates the collapses from Section 4.[4] We will use the results of Section 3 to show that this variant of Prikry forcing also satisfies the forcing theorem, and then show that this notion of forcing destroys replacement, however preserves separation to its generic set-extensions, thus answering [HKS, Question 9.2].

We will work over a very particular ground model $\mathbb{M}$, assuming the consistency of a measurable cardinal.[5] This model is almost the same as the one constructed in [HKS, Section 7]. We will nevertheless sketch its construction for the benefit of the reader.

Let $\tilde M$ be a countable transitive model of ZFC plus GCH (the additional GCH assumption will be the only difference to the model constructed in [HKS]), in which κ is a measurable cardinal and let U be a normal ultrafilter on κ in $\tilde M$. Let $M=V_\kappa^{\tilde M}$ and let $<$ be a wellorder of M in $\tilde M$. We will represent subsets of κ that are $\mathcal{L}_{\kappa,\kappa}$-definable over M with less than κ many additional predicates by codes in M. For a sequence $\langle P_i\mid i<\lambda\rangle$ of predicates, we use codes of the form $\langle\varphi,p\rangle$ for an $\mathcal{L}_{\kappa,\kappa}(\langle P_i\mid i<\lambda\rangle)$-formula φ with $\lambda<\kappa$ and $p\in M$ to describe $A(\varphi,p)=\{\alpha<\kappa\mid M\models\varphi(\alpha,p)\}$, and we will identify $\langle\varphi,p\rangle$ and $A(\varphi,p)$. We also let $\mathcal{L}_{\kappa,\kappa}(\langle P_i\mid i<\lambda\rangle)$ denote the collection of subsets of M that are $\mathcal{L}_{\kappa,\kappa}(\langle P_i\mid i<\lambda\rangle)$-definable over M.

We now define a sequence $\langle T_i\mid i<\mathrm{Ord}^M\rangle$ of truth predicates, a sequence $\langle U_i^*\mid i<\mathrm{Ord}^M\rangle$ of ultrafilters and a sequence of languages, and the corresponding definable subsets of M, $\langle\mathcal{L}^i_{\kappa,\kappa},\mathcal{L}^{i+}_{\kappa,\kappa}\mid i<\mathrm{Ord}^M\rangle$ as follows. Having defined the sequences $\langle T_j\mid j<i\rangle$ and $\langle U_j^*\mid j<i\rangle$, let $\mathcal{L}^i_{\kappa,\kappa}=\mathcal{L}_{\kappa,\kappa}(M,<,\langle T_j\mid j<i\rangle,\langle U_j^*\mid j<i\rangle)$ and let T_i be an M-truth predicate for $\mathcal{L}^i_{\kappa,\kappa}$-formulas. Having

[4] The coding will ensure that the counterexample to replacement that was provided by the generic filter in [HKS, Section 7] now becomes definable.

[5] In fact, the following construction can be performed starting from an assumption that is consistencywise strictly weaker than a measurable cardinal; we will elaborate on this in Section 6. This may be seen as a major difference to standard Prikry forcing.

defined $\langle T_j \mid j \leq i\rangle$ and $\langle U^*_j \mid j < i\rangle$, let $\mathcal{L}^{i+}_{\kappa,\kappa} = \mathcal{L}_{\kappa,\kappa}(M, <, \langle T_j \mid j \leq i\rangle, \langle U^*_j \mid j < i\rangle)$ and let $U^*_i = \{X \in U \mid X \in \mathcal{L}^{i+}_{\kappa,\kappa}\}$. Let $U^* = \bigcup_{i<\mathrm{Ord}} U^*_i$ and let $\mathcal{L}^{\mathrm{Ord}}_{\kappa,\kappa} = \bigcup_{i<\mathrm{Ord}^M} \mathcal{L}^i_{\kappa,\kappa}$. Our ground model will be

$$\mathbb{M} = \left\langle M, \mathrm{Def}\left(M, <, \langle T_i \mid i < \mathrm{Ord}^M\rangle, \langle U^*_i \mid i < \mathrm{Ord}^M\rangle\right)\right\rangle \models \mathsf{GBC}.$$

We will cite two results from [HKS] concerning the model $\mathbb{M}$, that we will make use of in our proof later on.

Lemma 5.1. [HKS, Lemma 7.1] *U^* is $<\kappa$-complete over M. For any $i \in \mathrm{Ord}^M$, U^*_i is $<\kappa$-complete over M and the sequence $\langle U^*_i \mid i < \mathrm{Ord}^M\rangle$ obeys some form of normality over M, namely for any $i \in \mathrm{Ord}^M$, whenever $\langle\langle\varphi_\alpha, p_\alpha\rangle \mid \alpha < \kappa\rangle \in \mathcal{L}^i_{\kappa,\kappa}$ is a sequence of pairs consisting of an $\mathcal{L}^i_{\kappa,\kappa}$-formula φ_α and a parameter $p_\alpha \in M$ such that $A(\varphi_\alpha, p_\alpha) \in U^*_i$ for every $\alpha < \kappa$, then the diagonal intersection $\triangle_{\alpha<\kappa} A(\varphi_\alpha, p_\alpha) \in U^*_i$.* □

The following lemma is provided in [HKS] only in the case that $\lambda = 2$, however the proof for arbitrary $\lambda \in \mathrm{Ord}^M$ works just the same, replacing each occurence of 2 by λ.

Lemma 5.2. [HKS, Lemma 7.2] *If $i \in \mathrm{Ord}^M$, $n \geq 1$ is a natural number, $\lambda \in \mathrm{Ord}^M$ and $h\colon \kappa^n \to \lambda$, $h \in \mathcal{L}^{i+}_{\kappa,\kappa}$, then there is $A \in U^*_{i+n-1}$ homogeneous for h, i.e. for any $t_0, t_1 \in [A]^n$, $h(t_0) = h(t_1)$.* □

We now define the forcing notion that we will use to obtain our desired result.

Definition 5.3. Let $\mathbb{P}$ be the notion of class forcing for $\mathbb{M}$ whose conditions are quadruples $\langle s, c, \varphi, p\rangle$ such that

(a) $s \in \mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$,
(b) c is a condition in C_s, that is the product of $\mathrm{Add}(\aleph_{\eta+1}, \aleph_{\eta+3})$ for $\eta \in s^{\mathrm{Ord}}$; we view c as a sequence of length $\max(s^{\mathrm{Ord}}) + 1$ with $c(\eta) \in \mathrm{Add}(\aleph_{\eta+1}, \aleph_{\eta+3})$ for $\eta \in s^{\mathrm{Ord}}$, and $c(\eta)$ trivial otherwise,
(c) φ is an $\mathcal{L}^{\mathrm{Ord}}_{\kappa,\kappa}$-formula with parameter $p \in M$,
(d) $A(\varphi, p) \in U^*$ and
(e) $\max s^{\mathrm{Ord}} < \min A(\varphi, p)$,

equipped with the ordering $\langle t, d, \psi, q\rangle \leq \langle s, c, \varphi, p\rangle$ iff

(1) t extends s in the ordering of $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$,
(2) $t^{\mathrm{Ord}} \setminus s^{\mathrm{Ord}} \subseteq A(\varphi, p)$,
(3) $A(\psi, q) \subseteq A(\varphi, p)$ and
(4) $d \leq_{C_t} c$, where $\leq_{C_t}$ denotes the usual ordering of C_t. [6]

As in the case of Prikry forcing, we can define a notion of *direct extension*, in fact we will define a notion of γ-direct extension for any ordinal γ. Namely, we let $\langle t, d, \psi, q\rangle \leq^\gamma \langle s, c, \varphi, p\rangle$ if $\langle t, d, \psi, q\rangle \leq \langle s, c, \varphi, p\rangle$, $t = s$ and $d \restriction \gamma = c \restriction \gamma$.

Observe that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$. Moreover, note that by the closure properties of $\mathrm{Add}(\aleph_{\eta+1}, \aleph_{\eta+3})$ for $\eta \geq \gamma$, $\langle \mathbb{P}, \leq^\gamma\rangle$ is $<\aleph_{\gamma+1}$-closed. Finally, we remark that $\mathbb{P}$ is not separative, because $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$ is not. Let φ^* and p^* be such that $A(\varphi^*, p^*) = \mathrm{Ord}^M$. For $n \in \omega$ and $\alpha \in \mathrm{Ord}^M$, let $\mathbb{P}^\alpha_n = \{\langle s, c, \varphi, p\rangle \in \mathbb{P} \mid s \in \mathrm{Col}^{n,\alpha}_{\geq}(\omega, \mathrm{Ord}), \varphi = \varphi^*, p = p^*\}$.

Lemma 5.4. *Every set added by $\mathbb{P}$ is added by a set-sized complete subforcing of $\mathbb{P}$, and $\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$.*

Proof. We check that $\mathbb{P}$ has the set reduction property, with the witnessing set-sized complete subforcings provided by the $\mathbb{P}^\alpha_n$ for $n \in \omega$ and $\alpha \in \mathrm{Ord}^M$, and thus the statement of the lemma follows using Lemma 3.6 and Lemma 3.9. To see that each $\mathbb{P}^\alpha_n$ is indeed a complete subforcing of $\mathbb{P}$, one proceeds very much like in the proof of Lemma 4.1, and we will thus omit the details of this argument.

[6] Formally, c may not be an element of C_t, but we may canonically identify it with one by extending it by a sequence of trivial conditions of appropriate length.

To verify the set reduction property for $\mathbb{P}$, let $r = \langle s, c, \varphi, p\rangle \in \mathbb{P}$ and let $A \subseteq \mathbb{P}$ be a set of conditions in M. For $a \in A$, we write a as $a = \langle s_a, c_a, \varphi_a, p_a\rangle$. Let $t \leq s$ in $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ such that for any $a \in A$ and any $i \in \mathrm{dom}(s_a)$, $\max t^{\mathrm{Ord}} > \mathrm{Ord}(s_a(i))$. By Lemma 5.1,

$$X = \left(\bigcap_{a\in A} A(\varphi_a, p_a)\right) \cap A(\varphi, p) \in U^*,$$

and there are ψ and q such that $X = A(\psi, q)$. Let $r' = \langle t, c, \psi, q\rangle$, let $n = \mathrm{dom}(t)$ and let $\mathbb{Q} = \mathbb{P}_n^{\max t^{\mathrm{Ord}}}$. We want to verify the set reduction property for $\mathbb{P}$ by showing that $(*)(A, r', \mathbb{Q})$ holds.

Suppose that $a \in A$. Clearly, $A(\psi, q) \subseteq A(\varphi_a, p_a)$. If $t \leq^* s_a$ in $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$ and $c \leq_{C_t} c_a$, then $r' \leq^*_{\mathbb{P}} a$, and we are done in this case. If $t \leq^* s_a$, however $c \perp_{C_t} c_a$, then $r' \perp_{\mathbb{P}} a$, and we are also done in this case. If $t \not\leq^* s_a$, then by our assumption on $\max t^{\mathrm{Ord}}$, t and s_a are incompatible in $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})^M$, thus also $r' \perp_{\mathbb{P}} a$ and we are done in this case as well. Finally, if $t \leq^* s_a$, $c \parallel_{C_t} c_a$ and $c \not\leq_{C_t} c_a$, let $v \in \mathbb{Q}^{\parallel r'}$. Let $a \upharpoonright n = \langle s_a \upharpoonright n, c_a, \varphi^*, p^*\rangle \in \mathbb{Q}^{\parallel r'}$. If $v \perp_{\mathbb{Q}^{\parallel r'}} a \upharpoonright n$, let $v' = v$, and let $v' \leq_{\mathbb{Q}^{\parallel r'}} v, a \upharpoonright n$ otherwise. Let $u \leq_{\mathbb{P}} r', v'$. In the first case, $u \perp_{\mathbb{P}} a$, and in the second case, $u \leq^*_{\mathbb{P}} a$. We have thus verified $(*)(A, r', \mathbb{Q})$ in each possible case. □

Given a $\mathbb{P}$-generic filter G over $\mathbb{M}$, let $g = \bigcup\{s^{\mathrm{Ord}} \mid \exists\langle c, \varphi, p\rangle\ \langle s, c, \varphi, p\rangle \in G\}$. An easy density argument shows that g is a cofinal function from ω to Ord^M, thus replacement fails in $\langle M[G], G\rangle$. The proof of Lemma 5.4 in fact shows the following.

Corollary 5.5. *$M[G] = \bigcup\{M[G_n^\alpha] \mid n < \omega, \alpha \in \mathrm{Ord}^M\}$, where G_n^α is the $\mathbb{P}_n^\alpha$-generic induced by G.* □

Claim 5.6. *$\mathbb{P}$ is cofinality-preserving and hence preserves all cardinals.*

Proof. Assume it is not. Let $\sigma \in M^{\mathbb{P}}$ name a witness, i.e. a function f from η to λ that is cofinal, where $\eta < \lambda$ are regular cardinals in M. By Corollary 5.5, f has a $\mathbb{P}_n^\alpha$-name for some $n \in \omega$ and $\alpha \in \mathrm{Ord}^M$. However, $\mathbb{P}_\alpha^n$ is cofinality-preserving under GCH by standard arguments about the relevant Cohen forcings, which yields a contradiction. □

Claim 5.7. *$M[G]$ satisfies the power set axiom, and whenever η is an infinite cardinal of M, $M[G] \models 2^\eta = \eta^{++}$ if and only if there is $\langle s, c, \varphi, p\rangle \in G$ and $\gamma \in s^{\mathrm{Ord}}$ with $\eta = \aleph_{\gamma+1}$.*

Proof. Let η be an infinite M-cardinal and let i be minimal so that $g(i-1) \geq \eta$. By Corollary 5.5, $\mathcal{P}(\eta)^{M[G]} = \bigcup\{\mathcal{P}(\eta)^{M[G_n^\alpha]} \mid n < \omega, \alpha \in \mathrm{Ord}^M\}$. But for every $n \in \omega$ and $\alpha \in \mathrm{Ord}^M$, $\mathcal{P}(\eta)^{M[G_n^\alpha]} \subseteq \mathcal{P}(\eta)^{M[G_i^\eta]}$, by the closure properties of the relevant instances of Cohen forcing. Thus $\mathcal{P}(\eta)^{M[G]} = \mathcal{P}(\eta)^{M[G_i^\eta]} \in M[G_i^\eta] \subseteq M[G]$. Essentially the same argument also yields the second statement of the claim. □

It is now immediate that g is definable over $M[G]$, and thus replacement fails in $M[G]$. We will be done once we have shown that separation holds in $M[G]$. To prove this, we will make use of the following homogeneity property of $\mathbb{P}$.

Lemma 5.8. *Working in M, assume that γ is an ordinal, $\dot{x}$ is a $\mathbb{P}$-name of rank at most γ, $\tau(\dot{x}) \in \mathcal{L}_\in^{\Vdash}$, $r = \langle s, c, \varphi, p\rangle$ forces that $\tau(\dot{x})$ holds, and $\max s^{\mathrm{Ord}} \geq \gamma$. Then $\bar{r} = \langle s, c \upharpoonright \gamma, \varphi, p\rangle$ forces that $\tau(\dot{x})$ holds.*

Proof. Assume for a contradiction that $\bar{r}$ does not force that $\tau(\dot{x})$ holds. Then there is an extension $\bar{u} = \langle t, d, \psi, q\rangle$ of $\bar{r}$ that forces $\neg\tau(\dot{x})$. We will obtain a contradiction by finding an automorphism π of $\mathbb{P}$ so that $\pi(\bar{u}) \parallel_{\mathbb{P}} r$ and so that π is the identity on conditions of rank less than γ. Let n be minimal such that $s(n) \geq \gamma$. Since $t \leq s$ in $\mathrm{Col}_{\geq}(\omega, \mathrm{Ord})$, $A(\psi, q) \subseteq A(\varphi, p)$ and $d \upharpoonright \gamma \leq_{C_{s\upharpoonright n}} c \upharpoonright \gamma$, we will let π be the identity on these components, and it thus suffices to find an automorphism π of $C := C_{t\setminus(s\upharpoonright n)}$ such that $\pi(d^{\geq\gamma}) \parallel_C c^{\geq\gamma}$, where $c^{\geq\gamma}$ is the sequence of the same length as c that is trivial below γ and identical to c at and above γ (similar for $d^{\geq\gamma}$). But the existence of such π is now an easy standard result about (a finite product of) Cohen forcings. □

We will also need the following Prikry property of $\mathbb{P}$.

Lemma 5.9. *Working in M, assume that γ is an ordinal, $\dot{x}$ is a $\mathbb{P}$-name of rank at most γ, and $\theta(\dot{x}) \in \mathcal{L}_\in^\Vdash$. Assume further that $r = \langle s, c, \varphi, p\rangle \in \mathbb{P}$ and that $\max(s^{\mathrm{Ord}}) \geq \gamma$. Then there is $u = \langle s, d, \psi, q\rangle \leq r$ that decides $\theta(\dot{x})$.*

Proof. Let n be least such that $s(n) \geq \gamma$. Define a function $h\colon [A(\varphi, p)]^{<\omega} \to \mathcal{P}(C_{s\restriction n})$ as follows:

$$h(t) = \{d \in C_{s\restriction n} \mid d \leq_{C_{s\restriction n}} c \restriction \gamma \text{ and for some } \psi \text{ and } q,\ \langle s^{\mathrm{Ord}} \cup t, d, \psi, q\rangle \Vdash_{\mathbb{P}} \theta(\dot{x})\}.$$

By Lemma 5.2, using that $\mathcal{P}(C_{s\restriction n}) \in M \models \mathsf{AC}$, there are φ' and p' such that $A(\varphi', p') \subseteq A(\varphi, p)$ is homogeneous for h, with $A(\varphi', p') \in U^*$. Pick a condition $u' = \langle s^{\mathrm{Ord}} \cup t, d, \psi, q\rangle \leq_{\mathbb{P}} \langle s, c, \varphi', p'\rangle$, with $t \in [\kappa]^{<\omega}$, that decides $\theta(\dot{x})$. By Lemma 5.8, we may assume that $d \in C_{s\restriction n}$. We claim that $u = \langle s, d, \psi, q\rangle$ decides $\theta(\dot{x})$ in the same way, and is thus as desired.

Assume for a contradiction that this is not the case. Then there is $u^0 = \langle s^{\mathrm{Ord}} \cup t', d', \psi', q'\rangle \leq_{\mathbb{P}} u$, with $t' \in [\kappa]^{<\omega}$, that decides $\theta(\dot{x})$ differently. By possibly strengthening either, we may assume that t and t' have the same cardinality. By Lemma 5.8, we may assume that $d' \in C_{s\restriction n}$. Note that $u^1 = \langle s^{\mathrm{Ord}} \cup t, d', \psi', q'\rangle \leq u'$. Since $h(t) = h(t')$ by homogeneity of $A(\varphi', p')$ however, u^0 and u^1 decide $\theta(\dot{x})$ in the same way, a contradiction. □

The following key result could be described as a *Prikry reduction property.*

Lemma 5.10. *Working in M, assume that γ is an ordinal, $\dot{x}$ is a $\mathbb{P}$-name of rank at most γ, and $\theta(\dot{x}) \in \mathcal{L}_\in^\Vdash$. Assume further that $r = \langle s, c, \varphi, p\rangle \in \mathbb{P}$ and that $\max(s^{\mathrm{Ord}}) \geq \gamma$. Let n be least such that $s(n) \geq \gamma$. Then there is $u = \langle s, d, \psi, q\rangle \leq^\gamma r$ that* reduces *$\theta(\dot{x})$* to *$\mathbb{Q} = \mathbb{P}^\gamma_n$, in the sense that there is a maximal antichain A in $\mathbb{Q}$ below $r \restriction n = \langle s \restriction n, c \restriction \gamma, \varphi^*, p^*\rangle$, such that for every $a = \langle s \restriction n, c_a, \varphi^*, p^*\rangle \in A$, the greatest lower bound $u \wedge_{\mathbb{P}} a = \langle s, c_a \wedge d, \psi, q\rangle$ of u and a decides $\theta(\dot{x})$.*

Proof. Note that the forcing $\mathbb{Q}$ below $\langle s \restriction n, \emptyset, \varphi^*, p^*\rangle$ has the $\aleph_{\gamma+1}$-cc, by our choice of n and by standard arguments about the instances of Cohen forcing involved. We will build a decreasing sequence $\langle r_\xi \mid \xi < \zeta\rangle$ of some length $\zeta < \aleph_{\gamma+1}$ of γ-direct extensions of r in $\mathbb{P}$, together with a sequence $\langle a_\xi \mid \xi < \zeta\rangle$ of incompatible conditions in $\mathbb{Q}$ below $r \restriction n$ as follows. We write each r_ξ as $r_\xi = \langle s, c_\xi, \varphi_\xi, p_\xi\rangle$. Given $\langle r_\rho \mid \rho < \xi\rangle$ and $\langle a_\rho \mid \rho < \xi\rangle$ for some $\xi < \aleph_{\gamma+1}$, let $r'_\xi = \langle s, c'_\xi, \varphi'_\xi, p'_\xi\rangle$ be the greatest lower bound of $\langle r_\rho \mid \rho < \xi\rangle$, that is a γ-direct extension of r. If possible, let $a_\xi = \langle s \restriction n, c^\xi_a, \varphi^*, p^*\rangle$ be a condition in $\mathbb{Q}$ below $r \restriction n$ that is incompatible to each element of $\{a_\rho \mid \rho < \xi\}$, and so that for some γ-direct extension r_ξ of r'_ξ, we have that $a_\xi \wedge_{\mathbb{P}} r_\xi = \langle s, c^\xi_a \wedge_{C_s} c_\xi, \varphi_\xi, p_\xi\rangle$ decides $\theta(\dot{x})$. If this is not possible, let the construction terminate and let $\zeta = \xi$.

Claim 5.11. *If the above construction terminates at stage ξ, then this is because $\{a_\rho \mid \rho < \xi\}$ is a maximal antichain in $\mathbb{Q}$ below $r \restriction n$.*

Proof. Assume that at stage ξ in our construction, there is a_ξ in $\mathbb{Q}$ below $r \restriction n$, that is incompatible to every a_ρ for $\rho < \xi$. We will be done if we can show that there is $a \leq_{\mathbb{Q}} a_\xi$ and a γ-direct extension r_ξ of r'_ξ such that $a \wedge_{\mathbb{P}} r_\xi$ decides $\theta(\dot{x})$. This amounts to showing that there is an extension $\langle s, d, \psi, q\rangle$ of $a_\xi \wedge_{\mathbb{P}} r'_\xi = \langle s, c^\xi_a \wedge_{C_s} c'_\xi, \varphi'_\xi, p'_\xi\rangle$ that decides $\theta(\dot{x})$. But this is exactly the statement of Lemma 5.9. □

Note that by the $\aleph_{\gamma+1}$-cc of $\mathbb{Q}$ below $r \restriction n$, the above construction has to terminate at some $\zeta < \aleph_{\gamma+1}$. Let u be the greatest lower bound of $\langle r_\rho \mid \rho < \zeta\rangle$, that is a γ-direct extension of r. Then u is as desired, as witnessed by $A = \{a_\xi \mid \xi < \zeta\}$, noting that $u \wedge_{\mathbb{P}} a_\xi \leq_{\mathbb{P}} r_\xi \wedge_{\mathbb{P}} a_\xi$ for every $\xi < \zeta$. □

Theorem 5.12. *If G is $\mathbb{P}$-generic over $\mathbb{M}$, separation holds in the generic set-extension $M[G]$.*

Proof. Note that if $X \in M[G]$, then $X \in M[G^\alpha_n]$ for some $n < \omega$ and $\alpha \in \mathrm{Ord}^M$, hence there is some M-cardinal λ and a bijection between λ and X in $M[G^\alpha_n] \subseteq M[G]$. It thus suffices to show that whenever $r \in \mathbb{P}$, $\dot{x}$ is a $\mathbb{P}$-name, $\varphi \in \mathcal{L}_\in$ and $\lambda < \kappa$, then there is $u \leq r$ forcing that $\{\alpha < \lambda \mid \varphi(\alpha, \dot{x})\}$ has a $\mathbb{P}^\gamma_n$-name for some $n < \omega$ and some $\gamma \in \mathrm{Ord}^M$. We denote first-order formulae in the forcing language of $\mathbb{P}$ as $\mathcal{L}_\in^\Vdash$-formulae. If we write $\theta(\dot{x}) \in \mathcal{L}_\in^\Vdash$, this additionally indicates that $\dot{x}$ is the only $\mathbb{P}$-name that appears in $\theta(\dot{x})$.

Let γ be the maximum of the rank of $\dot{x}$ and of λ. By possibly extending $r = \langle s, c, \varphi, p\rangle$, we may assume that $\max s^{\mathrm{Ord}} \geq \gamma$. Let n be least such that $s(n) \geq \gamma$. Now we build a decreasing sequence $\langle r_\alpha \mid \alpha \leq \lambda\rangle$ of γ-direct extensions of r with $r_0 = r$. Given r_α and using Lemma 5.10, let $r_{\alpha+1} \leq_\gamma r_\alpha$ reduce "$\varphi(\alpha, \dot{x})$" to $\mathbb{P}^\gamma_n$. Take greatest lower bounds at limit stages $\alpha \leq \lambda$. Then $u = r_\lambda$ forces that $\{\alpha < \lambda \mid \varphi(\alpha, \dot{x})\}$ has a $\mathbb{P}^\gamma_n$-name, as desired. □

6. On the consistency strength

We will show that the consistency strength of the assumption of the existence of a model $\mathbb{M}$ of the form that was used in the construction of the forcing in [HKS, Theorem 7.7] and of the forcing in Theorem 5.12 above lies strictly between a weakly compact cardinal and a measurable cardinal. This answers [HKS, Question 9.4].

For an upper bound, it was argued in [HKS, Section 7] that if κ is a measurable cardinal in a countable transitive model $\tilde{M}$ of ZFC, then there is a model $\mathbb{M} = (V_\kappa^{\tilde{M}}, \mathcal{C})$ of GBC with the desired properties. However the assumptions on $\mathbb{M}$ can be expressed as a Σ^1_1-statement over $V_\kappa^{\tilde{M}}$ and κ is Π^2_1-indescribable in $\tilde{M}$ by [Kan09, Proposition 6.5]. It follows that there is an inaccessible cardinal $\mu < \kappa$ in $\tilde{M}$ and a model $(V_\mu^{\tilde{M}}, \mathcal{D})$ satisfying our assumptions.

For a lower bound, we now show that there is a proper class of weakly compact cardinals in $\mathbb{M}$. We will use the notation introduced in Section 5.

Lemma 6.1. *Suppose that $\mathbb{M} = (M, \mathcal{C})$ is a transitive model of GBC with $\mathrm{Ord}^M = \kappa$ and $<\in \mathcal{C}$ is a well-order of M. Moreover, suppose that there are sequences $\langle T_i \mid i < \omega\rangle$ of truth predicates, $\langle U^*_i \mid i < \omega\rangle$ of filters and $\langle \mathcal{L}^i_{\kappa,\kappa}, \mathcal{L}^{i+}_{\kappa,\kappa} \mid i < \omega\rangle$ of languages (coded) in $\mathcal{C}$ such that for each $i < \omega$,*

$$\mathcal{L}^i_{\kappa,\kappa} = \mathcal{L}_{\kappa,\kappa}(M, <, \langle T_j \mid j < i\rangle, \langle U^*_j \mid j < i\rangle),$$
$$\mathcal{L}^{i^+}_{\kappa,\kappa} = \mathcal{L}_{\kappa,\kappa}(M, <, \langle T_j \mid j \leq i\rangle, \langle U^*_j \mid j < i\rangle),$$

T_i is an M-truth predicate for $\mathcal{L}^i_{\kappa,\kappa}$-formulas and U^*_i is a $<\kappa$-complete ultrafilter on $\mathcal{L}^{i^+}_{\kappa,\kappa}$ which satisfies normality with respect to $\mathcal{L}^i_{\kappa,\kappa}$-definable regressive functions, or equivalently with respect to $\mathcal{L}^i_{\kappa,\kappa}$-definable diagonal intersections.[7] *Finally, suppose that $\mathcal{C}$ is the collection of all subsets of M that are $\mathcal{L}^i_{\kappa,\kappa}$-definable over M for some $i < \omega$. Then there is a proper class of weakly compact cardinals in M.*

Proof. We work in $\mathbb{M}$. For each $i < \omega$, let

$$N_i = \mathrm{Ult}^{\mathcal{L}^i_{\kappa,\kappa}}(M, U^*_i)$$

denote the ultrapower of M with respect to $\mathcal{L}^i_{\kappa,\kappa}$-definable functions $f\colon \kappa \to M$.

Since $<$ is a well-order of M, Łos' theorem holds for each of the above ultrapowers by the same proof as the classical Łos' theorem. Since each U^*_i is $<\kappa$-complete, each N_i is well-founded, hence we identify N_i with its transitive collapse for all $i < \omega$. Note moreover that by $<\kappa$-completeness again, the corresponding elementary embeddings from M to N_i are given by the identity map for every $i < \omega$. Since U^*_i is normal and by Łos' theorem, $[\mathrm{id}]_{U^*_i} = \kappa$ in N_i for all $i < \omega$.

The next claim relates our assumption to that of κ-powerset preserving embeddings, which are used in [Git11, Proposition 2.8] to characterize Ramsey-like cardinals.

Claim 6.2. *$\mathcal{C} \cap \mathcal{P}(\kappa) = \bigcup_{i\in\omega} N_i \cap \mathcal{P}(\kappa)$.*

Proof. Assume first that $X \subseteq \kappa$ is in $\mathcal{C}$. Let $f\colon \kappa \to M$ be defined by setting $f(\alpha) = X \cap \alpha$ for every $\alpha < \kappa$. Then $[f]_{U^*_0} = X$ by normality and by Łos' theorem.

Assume now that $X \subseteq \kappa$ is an element of N_i for some $i < \omega$. Since T_{i+1} is an $\mathcal{L}^{i+1}_{\kappa,\kappa}$ truth predicate, we have a representation of N_i and its element relation in $\mathcal{C}$, by considering equivalence classes of $\mathcal{L}^i_{\kappa,\kappa}$-definable functions from κ to M. Assume that $X = [f]_{U^*_i}$. Letting c_α denote the constant function with domain κ and value α, $[c_\alpha]_{U^*_i} = \alpha$ for every $\alpha < \kappa$. Hence making use of the above representation of N_i in $\mathcal{C}$ and the closure of $\mathcal{C}$ under first order definability, $\{\alpha < \kappa \mid \alpha \in X\}$ is an element of $\mathcal{C}$. □

Claim 6.3. *Every club $C \in \mathcal{L}^i_{\kappa,\kappa}$ in κ is an element of U^*_i.*

[7] The precise condition is stated in Lemma 5.1.

Proof. Suppose that $\kappa \setminus C \in U_i^*$ for some club C in κ. Let $f\colon \kappa \setminus C \to \kappa$, $f(\alpha) = \max(C \cap \alpha)$. Since f is regressive and U_i^* is normal, $f^{-1}(\alpha) \in U_i^*$ for some $\alpha < \kappa$. The definition of f implies that $\max(C) = \alpha$, contradicting that C is unbounded in κ. □

Claim 6.4. *κ has the tree property in $\mathbb{M}$, i.e. there are no κ-Aronszajn trees in $\mathbb{M}$.*

Proof. Suppose for a contradiction that T is an $\mathcal{L}^i_{\kappa,\kappa}$-definable κ-Aronszajn tree in $\mathbb{M}$ with domain κ and tree order $<_T$. Suppose that C is the club in κ consisting of the ordinals $\alpha < \kappa$ with $\mathrm{Lev}_{<\alpha}(T) = \bigcup_{\bar{\alpha}<\alpha} \mathrm{Lev}_{\bar{\alpha}}(T) = \alpha$. Then $C \in U_0^*$ by Claim 6.3. Let $f_b\colon \kappa \to M$, where $f_b(\alpha)$ is the $<$-least branch through $t_\alpha = (\alpha, <_T \restriction \alpha)$ of length $\mathrm{ht}(t_\alpha)$ for all $\alpha \in C$, and $f(\alpha) = 0$ otherwise. By Łos' theorem, $[f_b]_{U_i^*} \restriction \alpha$ is a cofinal branch through t_α in N_i for every $\alpha \in C$. Thus $[f_b]_{U_i^*}$ is a cofinal branch through T in N_i. Since T_{i+1} is a truth definition for $\mathcal{L}^{i+}_{\kappa,\kappa}$, the ultrapower N_i is definable over $\mathbb{M}$, and hence there is a cofinal branch through T in $\mathbb{M}$. This contradicts the assumption that T is a κ-Aronszajn tree in $\mathbb{M}$. □

Claim 6.5. *The set A of inaccessible cardinals in M is an element of U_0^*.*

Proof. Since $V_\kappa^{N_0} = M$ and $V_{\kappa+1}^{N_0} \subseteq \mathcal{C}$ by Claim 6.2, κ is inaccessible in N_0. Since $[\mathrm{id}]_{U_0^*} = \kappa$, $A \in U_0^*$ by Łos' theorem.

□

Claim 6.6. *The set B of weakly compact cardinals in M is an element of U_0^*.*

Proof. Suppose that $\kappa \setminus B \in U_0^*$. Then the set of inaccessible non-weakly compact cardinals below κ is an element of U_0^* by Claim 6.5. Let $f\colon \kappa \to M$, where $f(\alpha)$ is the $<$-least κ-Aronszajn tree order on α, if α is inaccessible and not weakly compact, and $f(\alpha) = 0$ otherwise. Then $<_T := [f]_{U_0^*}$ is a tree order with domain κ, by normality and by Łos' theorem for N_0. We have $<_T = [f]_{U_0^*} = [f]_{U_i^*}$ for all $i < \omega$, since for all $\alpha, \beta < \kappa$, $(\alpha, \beta) \in [f]_{U_i^*}$ if and only if $\{\gamma < \kappa \mid (\alpha, \beta) \in f(\gamma)\} \in U_i^*$. This last statement does not depend on i, since the filters U_i form a $\subseteq$-increasing chain. By Łos' theorem, $(\kappa, <_T)$ is a κ-Aronszajn tree in N_i for all $i < \omega$. Therefore $(\kappa, <_T)$ is a κ-Aronszajn tree in $\mathbb{M}$ by Claim 6.2, contradicting Claim 6.4. □

This completes the proof of Lemma 6.1. □

7. Questions

The construction for the proof of Theorem 5.12 takes place over a model of GBC with many (non-definable) class predicates. We would thus like to ask the following slight variant of [HKS, Question 9.3], namely whether a similar result could be obtained starting over a first order model of ZFC.

Question 7.1. *Does separation imply replacement in generic set-extensions of first order models by definable class forcing that satisfies the forcing theorem?*

For certain models, there is a partial positive answer. If a notion of class forcing $\mathbb{P}$ which satisfies the forcing theorem is definable over a model of V = L, and there is a cofinal function from some $\lambda \in \mathrm{Ord}^M$ to Ord^M that is definable over $M[G]$,[8] then separation fails in $M[G]$. This is a direct consequence of [HKS, Lemma 5.1].

The consistency strength of the assumption of the existence of a model $\mathbb{M}$ as in the statement of Lemma 6.1, such as was used in the proof of Theorem 5.12, was shown in Section 6 to lie strictly between a weakly compact and a measurable cardinal. Thus the following question is immediate.

Question 7.2. *What is the consistency strength of the existence of a model $\mathbb{M}$ as in the statement of Lemma 6.1?*

Moreover, it is open whether the conclusions of [HKS, Theorem 7.7] and Theorem 5.12 have any large cardinal strength.

Question 7.3. *Does the existence of a generic extension for class forcing of a model of* GBC *which satisfies separation, but not replacement, imply the consistency of a weakly compact cardinal?*

[8]Note that this is a particular case of a failure of replacement in $M[G]$.

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